Extension of Nelson's Stochastic Quantization to Thermal Situation in Thermo Field Dynamics

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Abstract. We present an extension of Nelson's stochastic quantum mechanics to thermal situation. Utilizing the formulation of Thermo Field Dynamics (TFD) with doubled degree of freedom (non-tilde and tilde), we set up the stochastic equations and Nelson-Newton equations for tilde and non-tilde particles which reproduce the TFD-type Schrödinger equation, equivalent to the Liouville-von Neumann equation. In our formalism, the drift terms of the stochastic equations have the temperature dependence and the thermal fluctuation is induced through the quantum correlations of the non-tilde and tilde particles. As an example of application of our formalism, the position-momentum uncertainty relation at finite temperature is visualized.

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1. Introduction

The stochastic quantum mechanics was introduced by Nelson in 1966 [1]. There the motion of a particle in a microscopic world, governed by quantum theory, is described by the hypothetical stochastic processes, as the Brownian motion. The particle, obeying the classical equation of motion, is affected by hypothetical random noise. In Nelson's theory, one sets up the separate stochastic equations in forward and backward time directions, so that the temporal reversibility of microscopic motion is restored as a whole. Actually, from the set of the forward and backward stochastic equations, and the Newton type of the equation, called the Nelson-Newton equation, follows the reversible Schrödinger equation.

According to the Nelson's theory, we have an ensemble of an infinite number of classical trajectories (sample paths), and the average of any observable over them is equal to its expectation value, given by the wave function in quantum mechanics. But the Nelson's theory offers us a wider scope, i.e., we can calculate probability distributions and expectation values of physical quantities which can not be represented by operators, such as a time parameter. As an example, we refer to the tunneling time, arrival one and presence one of a particle in some region [2, 3, 4, 5].

After Nelson presented his theory, the extensions of the Nelson's approach were examined by several authors. A stochastic quantization based on stochastic action [6, 7] were introduced in 1980s. The generalization of Nelson's approach to interacting many particles were performed in Refs. [8, 9].

The applications of the Nelson's stochastic quantum mechanics to dissipative system were investigated in Refs. [10, 11, 12, 13]. For example, Ruggiero and Zannetti proposed a stochastic description of the harmonic oscillator for the ground state in equilibrium with a thermal bath [11, 12]. Using a coherent state of the harmonic oscillator, they introduce the classical solution and its quantum fluctuation which obeys the Nelson's stochastic quantum mechanics. Introducing the temperature dependent random noise to the classical equation, they constructed the diffusion process with thermal mixture for the harmonic oscillator. Their approach describes the dissipative process with quantum fluctuation and reduces to the Nelson's approach at zero temperature. But their method is restricted to the harmonic oscillator and it is not clear whether their method fulfills the properties described by quantum statistical mechanics, such as position-momentum uncertainty relation at finite temperature [14, 15].

The purpose of this paper is to extend the Nelson's stochastic quantum mechanics to thermal situation in framework of Thermo Field Dynamics (TFD) [16]. The average of a thermal mixed state is replaced by that of a pure state in the TFD formulation at the price of doubling each degree of freedom: a new tilde degree of freedom is introduced to each original non-tilde one, and the state vector space is a direct product of non-tilde and tilde state vector spaces. In other words, thermal fluctuations are realized as quantum fluctuations through the correlation between non-tilde and tilde sectors. Since in the Schrödinger picture of TFD the dynamical evolution is described

by the Schrödinger equation with doubled degrees of freedom which is equivalent to the Liouville-von Neumann equation for the density matrix, we attempt to establish the Nelson's stochastic description to this TFD-type Schrödinger equation.

This paper is organized as follows. In Sec. 2, we briefly review the Nelson's stochastic quantum mechanics and its equivalence to the Schrödinger equation. We introduce the TFD formalism and show its relation to the density matrix formalism in Sec. 3. Then we rewrite the TFD-type Schrödinger equation into the dynamical and kinematical equations , which will be derived from the Nelson's stochastic quantization. Section 4 is a main part of the paper, where we introduce the stochastic equations and Nelson-Newton equations for the non-tilde and tide particles. The set of these equations turns out to be equivalent to the TFD-type Schrödinger equation. In Sec. 5, we apply our formalism to a harmonic oscillator and show the position-momentum uncertainty relation at finite temperature. Section 6 is devoted to summary.

2. Nelson's stochastic quantum mechanics

In this section, we briefly review the Nelson's stochastic quantum mechanics and show its equivalence to the Schrödinger equation. Here, for simplicity, we consider the one-body Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{x},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x},t)\right)\Psi(\boldsymbol{x},t)\,, \tag{1}$$

with a mass m and an external potential $V(\boldsymbol{x},t)$. When we rewrite the wave function Ψ into the form $\Psi = e^{R+iS}$, where R and S are real functions, and define the current and osmotic velocities as

$$\boldsymbol{v} = \frac{\hbar}{m}\nabla S\,, \tag{2}$$

$$\boldsymbol{u} = \frac{\hbar}{m}\nabla R\,, \tag{3}$$

the gradient of equation (1) is transformed into the two equations, called the dynamical and kinematical equations, respectively,

$$\frac{\partial \boldsymbol{v}}{\partial t} = \frac{\hbar}{2m}\nabla^2\boldsymbol{u} + \boldsymbol{u}\cdot(\nabla\boldsymbol{u}) - \boldsymbol{v}\cdot(\nabla\boldsymbol{v}) - \frac{1}{m}\nabla V\,, \tag{4}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{\hbar}{2m}\nabla^2\boldsymbol{v} - \nabla(\boldsymbol{u}\cdot\boldsymbol{v})\,. \tag{5}$$

It can be shown as below that the Nelson's stochastic quantum mechanics leads to equations (4) and (5).

The first assumption of the Nelson's stochastic quantization is the stochastic differential equations for the particle position $\boldsymbol{x}(t)$. Since a single stochastic equation describes an irreversible process, we set up a pair of stochastic equations for forward and backward time evolutions separately, aiming at the reversible Schrödinger equation, explicitly

$$d\boldsymbol{x}(t) = \boldsymbol{b}(\boldsymbol{x}(t),t)dt + \sqrt{\frac{\hbar}{m}}d\boldsymbol{W} \qquad (d\boldsymbol{x}(t) = \boldsymbol{x}(t+dt) - d\boldsymbol{x}(t))\ , \tag{6}$$

for forward time evolution and

$$d\boldsymbol{x}(t) = \boldsymbol{b}_*(\boldsymbol{x}(t), t)dt + \sqrt{\frac{\hbar}{m}} d\boldsymbol{W}_* \qquad (d\boldsymbol{x}(t) = \boldsymbol{x}(t) - d\boldsymbol{x}(t - dt)) \,, \tag{7}$$

for backward one ($dt > 0$). Here $\boldsymbol{W}$ and $\boldsymbol{W}_*$ are the standard Wiener processes

$$E[dW_i(t)] = E[dW_{i*}(t)] = 0 \qquad i = x, y, z \,, \tag{8}$$

$$E[dW_i(t)dW_j(t)] = E[dW_{*i}(t)dW_{*j}(t)] = \delta_{ij} dt \,, \tag{9}$$

where $E[\cdots]$ means a sample average.

The second assumption is the Nelson-Newton equation of motion, defining the mean acceleration $\boldsymbol{a}(t)$ of random variables $\boldsymbol{x}(t)$ as follows:

$$m\boldsymbol{a}(t) = -\nabla V \,. \tag{10}$$

Here, the mean acceleration $\boldsymbol{a}(t)$ is defined as

$$\boldsymbol{a}(t) = \frac{1}{2}(DD_* + D_* D)\boldsymbol{x}(t) \,. \tag{11}$$

with the mean forward time derivative D,

$$Df(t) = \lim_{dt \to 0+} E\left[\frac{f(\boldsymbol{x}(t + dt)) - f(\boldsymbol{x}(t))}{dt} \,\middle|\, \boldsymbol{x}(t) \right] , \tag{12}$$

and the mean backward one D_*,

$$D_* f(t) = \lim_{dt \to 0+} E\left[\frac{f(\boldsymbol{x}(t)) - f(\boldsymbol{x}(t - dt))}{dt} \,\middle|\, \boldsymbol{x}(t) \right] , \tag{13}$$

where $E[\cdots|\boldsymbol{x}(t)]$ means the conditional expectation. From equations (6) and (7), the expression of the mean acceleration $\boldsymbol{a}(t)$ is obtained as

$$\begin{aligned} \boldsymbol{a}(t) = {} & \frac{\partial}{\partial t}\frac{(D + D_*)\boldsymbol{x}}{2} + \frac{1}{2}(D_*\boldsymbol{x} \cdot \nabla)D\boldsymbol{x} + \frac{1}{2}(D\boldsymbol{x} \cdot \nabla)D_*\boldsymbol{x} \\ & - \frac{\hbar}{2m}\nabla^2 \frac{(D - D_*)\boldsymbol{x}}{2} \,. \end{aligned} \tag{14}$$

Let us define the current and osmotic velocities as

$$\boldsymbol{v} = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{b}_*) = \frac{1}{2}(D + D_*)\boldsymbol{x} \,, \tag{15}$$

$$\boldsymbol{u} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{b}_*) = \frac{1}{2}(D - D_*)\boldsymbol{x} \,, \tag{16}$$

where we utilize the relations, $\boldsymbol{b} = D\boldsymbol{x}$ and $\boldsymbol{b}_* = D_*\boldsymbol{x}$. Then, substituting equation (14) into the Nelson-Newton equation of motion (10), we obtain the equation for the current velocity:

$$\frac{\partial}{\partial t}\boldsymbol{v} = \frac{\hbar}{2m}\nabla^2\boldsymbol{u} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \frac{1}{m}\nabla V \,. \tag{17}$$

The stochastic processes of the random variable $\boldsymbol{x}(t)$ in equations (6) and (7) are equivalently formulated by means of the distribution function $P(\boldsymbol{x}, t)$ which satisfy the Fokker-Planck equations,

$$\frac{\partial}{\partial t}P = -\nabla \cdot (\boldsymbol{b}P) + \frac{\hbar}{2m}\nabla^2 P \tag{18}$$

for forward time and

$$\frac{\partial}{\partial t}P = -\nabla \cdot (\boldsymbol{b}_* P) - \frac{\hbar}{2m}\nabla^2 P \tag{19}$$

for backward time. The sum of equations (18) and (19) is the continuity equation

$$\frac{\partial}{\partial t}P + \nabla \cdot (\boldsymbol{v} P) = 0\,, \tag{20}$$

while their difference yields the relation

$$\nabla \cdot \left\{ \boldsymbol{u} P - \frac{\hbar}{2m}\nabla P \right\} = 0\,. \tag{21}$$

Though the last equation implies $\boldsymbol{u}P - \frac{\hbar}{2m}\nabla P = \nabla \times \boldsymbol{C}$ with an arbitrary vector function $\boldsymbol{C}(\boldsymbol{x}, t)$, it is shown [17] that $\nabla \times \boldsymbol{C} = \mathbf{0}$, i.e.,

$$\boldsymbol{u} = \frac{\hbar}{2m}\nabla \ln P\,. \tag{22}$$

From equations (20) and (22), one derives

$$\frac{\partial}{\partial t}\boldsymbol{u} = \frac{\hbar}{2m}\frac{\partial}{\partial t}\nabla \ln P = -\frac{\hbar}{2m}\nabla(\nabla \cdot \boldsymbol{v}) - \nabla(\boldsymbol{u} \cdot \boldsymbol{v})\,. \tag{23}$$

Assuming that $\boldsymbol{v}$ is given by a gradient of a scalar function, we have the equation for the osmotic velocity as

$$\frac{\partial}{\partial t}\boldsymbol{u} = -\frac{\hbar}{2m}\nabla^2 \boldsymbol{v} - \nabla(\boldsymbol{u} \cdot \boldsymbol{v})\,, \tag{24}$$

using $\mathbf{0} = \nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}$.

Thus the equations of the current and osmotic velocities derived from Nelson's stochastic quantum mechanics, equations (17) and (24), are identical to equations (4) and (5) originating from the Schrödinger equation.

3. Thermo Field Dynamics

In this section, we introduce a formulation of Thermo Field Dynamics (TFD) in which the average over a density matrix is represented by a pure state average in the doubled Hilbert space. To each original operator A, called a non-tilde operator, we introduce a new independent one $\tilde{A}$, called a tilde one.

The eigenvalue problems for the non-tilde and tilde Hamiltonians are

$$H|u_n\rangle = E_n|u_n\rangle\,. \qquad |u_n\rangle \in \mathcal{H}\,, \tag{25}$$

$$\tilde{H}|\tilde{u}_n\rangle = E_n|\tilde{u}_n\rangle\,. \qquad |\tilde{u}_n\rangle \in \tilde{\mathcal{H}}\,, \tag{26}$$

respectively. We construct a pure state vector $|\Phi\rangle$ from a superposition of particular vectors $|u_n, \tilde{u}_n\rangle \in \hat{\mathcal{H}}$ with a common n in the doubled Hilbert space $\hat{\mathcal{H}} = \mathcal{H} \otimes \tilde{\mathcal{H}}$ with real coefficients f_n as

$$|\Phi\rangle = \sum_n f_n |u_n, \tilde{u}_n\rangle\,. \tag{27}$$

Then the expectation value of any non-tilde operator A for this $|\Phi\rangle$ becomes

$$\langle\Phi|A|\Phi\rangle = \sum_n f_n^2 \langle u_n|A|u_n\rangle = Tr[\rho A]\,, \tag{28}$$

$$\rho = \sum_n f_n^2 |u_n\rangle\langle u_n|\,, \tag{29}$$

where the ortho-normal condition $\langle u_n|u_m\rangle = \langle\tilde{u}_n|\tilde{u}_m\rangle = \delta_{nm}$ has been used. This way the expectation value for the pure state vector $|\Phi\rangle$ which is called a thermal vacuum is equivalent to the trace average over the density matrix ρ. If we set $f_n^{(\mathrm{eq})} = e^{-\frac{\beta E_n}{2}}/(\sum_n e^{-\beta E_n})^{1/2}$, where β is the inverse temperature, equation (28) is reduced to the trace average in thermal equilibrium. We may say that the TFD formalism represents the thermal fluctuation through the quantum correlation between non-tilde and tilde particles.

In TFD, the total Hamiltonian of the system is given by

$$\hat{H} = H - \tilde{H}\,, \tag{30}$$

and the Heisenberg operators $A(t)$ and $\tilde{A}(t)$ obey the Heisenberg equations

$$i\hbar\frac{d}{dt}A = [A, \hat{H}] = [A, H]\,, \tag{31}$$

$$i\hbar\frac{d}{dt}\tilde{A} = [\tilde{A}, \hat{H}] = -[\tilde{A}, \tilde{H}]\,. \tag{32}$$

Note that all the canonical commutation relations between non-tilde and tilde operators vanish. The minus sign in front of $\tilde{H}$ in equation (30) is crucial, which is required from the stability of the thermal vacuum $|\Phi\rangle$ in equation (27) including the thermal equilibrium [16]. The above equations imply that the tilde system corresponds to the time reversed one of the non-tilde particle.

The exchange of non-tilde and tilde operators is summarized in the tilde conjugation rules [16] in TFD:

$$(c_1A + c_2B)^{\sim} = c_1^*\tilde{A} + c_2^*\tilde{B} \qquad (c_i : \text{complex numbers}) \tag{33}$$

$$(\tilde{A})^{\sim} = A \tag{34}$$

$$(A^\dagger)^{\sim} = \tilde{A}^\dagger \tag{35}$$

$$(AB)^{\sim} = \tilde{A}\tilde{B}\,. \tag{36}$$

We note that the thermal vacuum is invariant under the tilde conjugation:

$$|\Phi\rangle^{\sim} = |\Phi\rangle\,, \quad \langle\Phi|^{\sim} = \langle\Phi|\,, \tag{37}$$

where $(c|u_n, \tilde{u}_m\rangle)^{\sim} = c^*|u_m, \tilde{u}_n\rangle$.

The time evolution of any state vector $|\Psi(t)\rangle$ in the Schrödinger picture is governed by the TFD-type Schrödinger equation:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = (H - \tilde{H})|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle\,. \tag{38}$$

For the thermal vacuum $|\Phi\rangle$, this equation reads as

$$0 = \hat{H}|\Phi\rangle\,, \tag{39}$$

implying that the thermal vacuum is a stationary state with zero eigenvalue. We require that the state vector $|\Psi(t)\rangle$ describing thermal situation should satisfy the tilde invariance

$$(|\Psi(t)\rangle)^{\sim} = |\Psi(t)\rangle\,. \tag{40}$$

The reason for this will be clear soon, in connection to the equivalence to the density matrix formalism.

As an example of using equation (38), one can imagine the case in which an isolated system is initially in a stationary thermal vacuum in equation (27) and a time-dependent external potential is switched on at $t = 0$. Then the total Hamiltonian is time-dependent, $\hat{H}(t) = H(t) - \tilde{H}(t)$ and the state vector is given by

$$|\Psi(t)\rangle = \sum_n f_n |u_n(t), \tilde{u}_n(t)\rangle\,, \tag{41}$$

where $|u_n(t)\rangle$ and $|\tilde{u}_n(t)\rangle$ obey the following equations

$$i\hbar\frac{d}{dt}|u_n(t)\rangle = H(t)|u_n(t)\rangle\,, \tag{42}$$

$$i\hbar\frac{d}{dt}|\tilde{u}_n(t)\rangle = -\tilde{H}(t)|\tilde{u}_n(t)\rangle\,, \tag{43}$$

under the initial conditions: $|u_n(t=0)\rangle = |u_n\rangle$ and $|\tilde{u}_n(t=0)\rangle = |\tilde{u}_n\rangle$. Note that the tilde invariance of the state vector $|\Psi(t)\rangle$ is preserved under the time evolution.

The TFD formalism has a one-to-one correspondence to the super-operator formalism for density matrix [18, 19]. In the Schrödinger picture of quantum statistical mechanics, the density matrix $\rho(t)$ obeys the Liouville-von Neumann equation

$$i\hbar\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t)\,, \tag{44}$$

where $\mathcal{L}$ is the Liouville operator:

$$\mathcal{L}\cdots = [H, \cdots]\,. \tag{45}$$

In the super-operator formalism, the density matrix is treated as a vector in a vector space on which an operator acts in two ways: an operation from the left-hand side of ρ as $A\rho$, and the other one from the right-hand side of ρ as ρA. We may rewrite right side operation as $\tilde{A}^{\dagger}\rho = \rho A$. This is the way how every operator is doubled and the operator A accompany its tilde conjugation $\tilde{A}$. Furthermore, rewriting the Liouville operator as $\mathcal{L}\rho = (H - \tilde{H})\rho$, we can recognize the form of the TFD Hamiltonian: $H - \tilde{H}$. The state vector $|\Psi\rangle$ in the doubled Hilbert space is equivalent to a density matrix ρ and its tilde invariance in equation (40) corresponds to the hermiticity of the density matrix $\rho^{\dagger} = \rho$. The correspondence between the super-operator formalism and TFD is as follows,

$$\rho \quad\Longleftrightarrow\quad |\Psi\rangle\,, \tag{46}$$

$$A\rho \quad\Longleftrightarrow\quad A|\Psi\rangle\,, \tag{47}$$

$$\rho A^{\dagger} \quad\Longleftrightarrow\quad \tilde{A}|\Psi\rangle\,, \tag{48}$$

$$\mathcal{L} \quad\Longleftrightarrow\quad \hat{H} = H - \tilde{H}\,, \tag{49}$$

$$i\hbar\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) \quad\Longleftrightarrow\quad i\hbar\frac{d}{dt}|\Psi(t)\rangle = (H - \tilde{H})|\Psi(t)\rangle\,. \tag{50}$$

Finally, we rewrite the TFD-type Schrödinger equation into the from of the dynamical and kinematical equations. In the coordinate representation ($\Psi(\boldsymbol{x},\tilde{\boldsymbol{x}},t) = \langle \boldsymbol{x},\tilde{\boldsymbol{x}}|\Psi(t)\rangle$), the TFD-type Schrödinger equation is given by

$$i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{x},\tilde{\boldsymbol{x}},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x},t) + \frac{\hbar^2}{2m}\tilde{\nabla}^2 - V(\tilde{\boldsymbol{x}},t)\right]\Psi(\boldsymbol{x},\tilde{\boldsymbol{x}},t)\,, \tag{51}$$

with the tilde invariance

$$\Psi(\boldsymbol{x},\tilde{\boldsymbol{x}},t) = (\Psi(\boldsymbol{x},\tilde{\boldsymbol{x}},t))^{\sim} = \Psi^*(\tilde{\boldsymbol{x}},\boldsymbol{x},t)\,. \tag{52}$$

Now, let us put the wave function $\Psi = e^{R+iS}$ and introduce the current and osmotic velocities as

$$\boldsymbol{v} = \frac{\hbar}{m}\nabla S\,, \quad \tilde{\boldsymbol{v}} = -\frac{\hbar}{m}\tilde{\nabla}S\,, \tag{53}$$

$$\boldsymbol{u} = \frac{\hbar}{m}\nabla R\,, \quad \tilde{\boldsymbol{u}} = \frac{\hbar}{m}\tilde{\nabla}R\,. \tag{54}$$

Then equation (51) is transformed into the dynamical and kinematical equations

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial t} &= \frac{\hbar}{2m}(\nabla^2 - \tilde{\nabla}^2)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla - \tilde{\boldsymbol{u}}\cdot\tilde{\nabla})\boldsymbol{u} \\ &\quad - (\boldsymbol{v}\cdot\nabla + \tilde{\boldsymbol{v}}\cdot\tilde{\nabla})\boldsymbol{v} - \frac{1}{m}\nabla(V - \tilde{V})\,,\end{aligned} \tag{55}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{\hbar}{2m}(\nabla^2 - \tilde{\nabla}^2)\boldsymbol{v} - \nabla(\boldsymbol{u}\cdot\boldsymbol{v} + \tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{v}})\,, \tag{56}$$

and their tilde conjugates. The notation $\tilde{V}$ stands for $V(\tilde{\boldsymbol{x}},t)$.

4. Nelson's Stochastic Quantization in Thermo Field Dynamics

In this section, we extend the Nelson's stochastic quantization to systems described by TFD. Our aim is to drive the dynamical and kinematical equations (55) and (56), equivalent to the Liouville-von Neumann equation, from the stochastic differential equations.

A naive extension of non-tilde system to non-tilde and tilde one is unsuccessful, since then the total Hamiltonian would be not $H - \tilde{H}$ but $H + \tilde{H}$. In order to accommodate the minus sign in front of $\tilde{H}$, we classify the four stochastic differential equations into the following two groups: one is a pair of those for $\boldsymbol{x}(t)$ in forward evolution and $\tilde{\boldsymbol{x}}(t)$ in backward one

$$d\boldsymbol{x}(t) = \boldsymbol{b}(\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t),t)dt + \sqrt{\frac{\hbar}{m}}d\boldsymbol{W}\,, (d\boldsymbol{x}(t) = \boldsymbol{x}(t+dt) - d\boldsymbol{x}(t))\;, \tag{57}$$

$$d\tilde{\boldsymbol{x}}(t) = \tilde{\boldsymbol{b}}_*(\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t),t)dt + \sqrt{\frac{\hbar}{m}}d\tilde{\boldsymbol{W}}_*\,, (d\tilde{\boldsymbol{x}}(t) = \tilde{\boldsymbol{x}}(t) - d\tilde{\boldsymbol{x}}(t-dt))\;, \tag{58}$$

and the other is a pair of those for $\boldsymbol{x}(t)$ in backward evolution and $\tilde{\boldsymbol{x}}(t)$ in forward one

$$d\boldsymbol{x}(t) = \boldsymbol{b}_*(\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t),t)dt + \sqrt{\frac{\hbar}{m}}d\boldsymbol{W}_*\,, (d\boldsymbol{x}(t) = \boldsymbol{x}(t) - d\boldsymbol{x}(t-dt))\;, \tag{59}$$

$$d\tilde{\boldsymbol{x}}(t) = \tilde{\boldsymbol{b}}(\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t),t)dt + \sqrt{\frac{\hbar}{m}}d\tilde{\boldsymbol{W}}\,, (d\tilde{\boldsymbol{x}}(t) = \tilde{\boldsymbol{x}}(t+dt) - d\tilde{\boldsymbol{x}}(t))\;. \tag{60}$$

Here $d\boldsymbol{W}$, $d\tilde{\boldsymbol{W}}_*$, $d\boldsymbol{W}_*$, $d\tilde{\boldsymbol{W}}$ are independent Wiener processes. The above grouping is suggested by the fact that the equation of the tilde particle is a time reversed one of the non-tilde particle in TFD.

A crucial step in our present formulation is to introduce new mean time derivatives by

$$\begin{aligned}&\bar{D}f(\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t),t)\\&=\lim_{dt\to 0+}E\left[\frac{f(\boldsymbol{x}(t+dt),\tilde{\boldsymbol{x}}(t),t+dt)-f(\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t-dt),t)}{dt}\,\Big|\,\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t)\right],\end{aligned} \tag{61}$$

$$\begin{aligned}&\bar{D}_*f(\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t),t)\\&=\lim_{dt\to 0+}E\left[\frac{f(\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t+dt),t)-f(\boldsymbol{x}(t-dt),\tilde{\boldsymbol{x}}(t),t-dt)}{dt}\,\Big|\,\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t)\right],\end{aligned} \tag{62}$$

The mean derivatives $\bar{D}$ and $\bar{D}_*$ are hybrids of the original D and D_*, and give the formulas for $f(\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t),t)$,

$$\bar{D}f=\frac{\partial f}{\partial t}+\boldsymbol{b}\cdot(\nabla f)+\tilde{\boldsymbol{b}}_*\cdot(\tilde{\nabla}f)+\frac{\hbar}{2m}\nabla^2 f-\frac{\hbar}{2m}\tilde{\nabla}^2 f\,, \tag{63}$$

$$\bar{D}_*f=\frac{\partial f}{\partial t}+\boldsymbol{b}_*\cdot(\nabla f)+\tilde{\boldsymbol{b}}\cdot(\tilde{\nabla}f)+\frac{\hbar}{2m}\nabla^2 f-\frac{\hbar}{2m}\tilde{\nabla}^2 f\,. \tag{64}$$

Then, utilizing the relations for any function $f(\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t))$,

$$\frac{d}{dt}E[f]=E[\bar{D}f]=E[\bar{D}_*f]\,, \tag{65}$$

we can obtain the Fokker-Planck equations for the distribution $P(\boldsymbol{x},\tilde{\boldsymbol{x}},t)$,

$$\frac{\partial P}{\partial t}=-\nabla\cdot(\boldsymbol{b}P)-\tilde{\nabla}\cdot(\tilde{\boldsymbol{b}}_*P)+\frac{1}{2}\frac{\hbar}{m}\left(\nabla^2-\tilde{\nabla}^2\right)P\,, \tag{66}$$

$$\frac{\partial P}{\partial t}=-\nabla\cdot(\boldsymbol{b}_*P)-\tilde{\nabla}\cdot(\tilde{\boldsymbol{b}}P)-\frac{1}{2}\frac{\hbar}{m}\left(\nabla^2-\tilde{\nabla}^2\right)P\,. \tag{67}$$

Here, we define the current and osmotic velocities of the non-tilde and tilde particles as

$$\boldsymbol{u}=\frac{1}{2}(\boldsymbol{b}-\boldsymbol{b}_*)\,,\quad \boldsymbol{v}=\frac{1}{2}(\boldsymbol{b}+\boldsymbol{b}_*)\,, \tag{68}$$

$$\tilde{\boldsymbol{u}}=\frac{1}{2}(\tilde{\boldsymbol{b}}-\tilde{\boldsymbol{b}}_*)\,,\quad \tilde{\boldsymbol{v}}=\frac{1}{2}(\tilde{\boldsymbol{b}}+\tilde{\boldsymbol{b}}_*)\,, \tag{69}$$

respectively. The difference of equations (66) and (67) gives us the relation:

$$\nabla\cdot\left(\boldsymbol{u}P-\frac{\hbar}{2m}\nabla P\right)-\tilde{\nabla}\cdot\left((\tilde{\boldsymbol{u}}P)-\frac{\hbar}{2m}\tilde{\nabla}P\right)=0\,. \tag{70}$$

The argument for deriving equation (22) applies here as well, so we have

$$\boldsymbol{u}=\frac{\hbar}{2m}\nabla\ln P\,,\qquad \tilde{\boldsymbol{u}}=\frac{\hbar}{2m}\tilde{\nabla}\ln P\,. \tag{71}$$

The sum of equations (66) and (67) is the continuity equation,

$$\frac{\partial}{\partial t}P+\nabla\cdot(\boldsymbol{v}P)+\tilde{\nabla}\cdot(\tilde{\boldsymbol{v}}P)=0\,. \tag{72}$$

From equations (71) and (72) follows the TFD-type kinematical equation (56) as

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{u} &= \frac{\hbar}{2m}\nabla\left(\frac{1}{P}\frac{\partial P}{\partial t}\right)\\ &= \frac{\hbar}{2m}\nabla\left\{-\frac{1}{P}\nabla\cdot(\boldsymbol{v}P) - \frac{1}{P}\tilde{\nabla}\cdot(\tilde{\boldsymbol{v}}P)\right\}\\ &= \nabla\left\{-\frac{\hbar}{2m}\nabla\cdot\boldsymbol{v} - \frac{\hbar}{2m}\tilde{\nabla}\cdot\tilde{\boldsymbol{v}} - \boldsymbol{v}\cdot\boldsymbol{u} - \tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{u}}\right\}\\ &= -\frac{\hbar}{2m}\left(\nabla^2 - \tilde{\nabla}^2\right)\boldsymbol{v} - \nabla(\boldsymbol{u}\cdot\boldsymbol{v} + \tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{v}})\,. \end{aligned} \tag{73}$$

In the last equality we have assumed that $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ are gradients of a single function $g(\boldsymbol{x}, \tilde{\boldsymbol{x}}, t)$ as $\boldsymbol{v} = \nabla g$ and $-\tilde{\boldsymbol{v}} = \tilde{\nabla} g$, so that

$$\nabla(\nabla\cdot\boldsymbol{v}) = \nabla^2\boldsymbol{v}\,, \qquad \nabla(\tilde{\nabla}\cdot\tilde{\boldsymbol{v}}) = -\nabla\tilde{\nabla}^2 g = -\tilde{\nabla}^2\boldsymbol{v}\,. \tag{74}$$

Next, we derive an expression of the mean acceleration $\boldsymbol{a}(t)$. Its natural candidate would be an average of the quantities $\bar{D}_*\bar{D}\boldsymbol{x}(t)$ and $\bar{D}\bar{D}_*\boldsymbol{x}(t)$, each of which can be computed as

$$\bar{D}_*\bar{D}\boldsymbol{x} = \frac{\partial\boldsymbol{b}}{\partial t} + (\boldsymbol{b}_*\cdot\nabla)\boldsymbol{b} + (\tilde{\boldsymbol{b}}\cdot\tilde{\nabla})\boldsymbol{b} - \frac{\hbar}{2m}(\nabla^2 - \tilde{\nabla}^2)\boldsymbol{b}\,, \tag{75}$$

$$\bar{D}\bar{D}_*\boldsymbol{x} = \frac{\partial\boldsymbol{b}_*}{\partial t} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}_* + (\tilde{\boldsymbol{b}}_*\cdot\tilde{\nabla})\boldsymbol{b}_* + \frac{\hbar}{2m}(\nabla^2 - \tilde{\nabla}^2)\boldsymbol{b}_*\,, \tag{76}$$

and their tilde conjugates. Then, the mean accelerations $\boldsymbol{a}$ and $\tilde{\boldsymbol{a}}$ are given by

$$\boldsymbol{a} = \frac{\partial\boldsymbol{v}}{\partial t} - \frac{\hbar}{2m}\left(\nabla^2 - \tilde{\nabla}^2\right)\boldsymbol{u} + (\boldsymbol{v}\cdot\nabla + \tilde{\boldsymbol{v}}\cdot\tilde{\nabla})\boldsymbol{v} - (\boldsymbol{u}\cdot\nabla - \tilde{\boldsymbol{u}}\cdot\tilde{\nabla})\boldsymbol{u}\,, \tag{77}$$

and its tilde conjugate. We require the following Nelson-Newton equations,

$$m\boldsymbol{a} = -\nabla(V - \tilde{V})\,, \tag{78}$$

$$m\tilde{\boldsymbol{a}} = \tilde{\nabla}(V - \tilde{V})\,. \tag{79}$$

Substituting equation (77) into the former equation, we obtain the TFD-type dynamical equation:

$$\begin{aligned}\frac{\partial\boldsymbol{v}}{\partial t} = \frac{\hbar}{2m}&\left(\nabla^2 - \tilde{\nabla}^2\right)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla - \tilde{\boldsymbol{u}}\cdot\tilde{\nabla})\boldsymbol{u}\\ &- (\boldsymbol{v}\cdot\nabla + \tilde{\boldsymbol{v}}\cdot\tilde{\nabla})\boldsymbol{v} - \frac{1}{m}\nabla(V - \tilde{V})\,. \end{aligned} \tag{80}$$

Consequently, if any solution of the TFD-type Schrödinger equation $\Psi(\boldsymbol{x}, \tilde{\boldsymbol{x}}, t) = e^{R+iS}$ is given, the drift terms in the stochastic equations are given by the gradients of R and S as

$$\boldsymbol{b} = \frac{\hbar}{m}\nabla(R + S)\,, \tag{81}$$

$$\boldsymbol{b}_* = -\frac{\hbar}{m}\nabla(R - S)\,, \tag{82}$$

$$\tilde{\boldsymbol{b}} = \frac{\hbar}{m}\tilde{\nabla}(R - S)\,, \tag{83}$$

$$\tilde{\boldsymbol{b}}_* = -\frac{\hbar}{m}\tilde{\nabla}(R + S)\,. \tag{84}$$

Thus we have the Nelson's stochastic quantum mechanics, equivalent to the TFD-type Schrödinger equation.

5. Application to Harmonic Oscillator and Generalized Uncertainty Relation

We apply the Nelson's stochastic quantum mechanics in the previous section to a particle in one dimensional harmonic oscillator. The TFD-type Schrödinger equation is given by

$$i\hbar\frac{\partial}{\partial t}\Psi(x,\tilde{x},t) = \left(H-\tilde{H}\right)\Psi(x,\tilde{x},t)\,, \tag{85}$$

$$H = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{\omega^2 m}{2}x^2\right)\,. \tag{86}$$

In thermal equilibrium, the wave function Ψ is given by

$$\Psi_{\rm eq}(x,\tilde{x}) = \sum_n \frac{e^{-\frac{\beta\hbar\omega n}{2}}}{Z(\beta)^{1/2}}u_n(x)u_n^*(\tilde{x})\,, \tag{87}$$

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{\omega^2 m}{2}x^2\right)u_n(x) = \hbar\omega\left(n+\frac{1}{2}\right)u_n(x)\,, \tag{88}$$

with $n=0,1,2,\cdots$. Using the Feynman kernel, defined by

$$\psi(x_1,t_1) = \int K(x_1,t_1;x_2,t_2)\psi(x_2,t_2)\,dx_2\,, \tag{89}$$

of harmonic oscillator

$$K(x_1,t_1;x_2,t_2) = \sqrt{\frac{m\omega}{2\pi\hbar\sin\omega(t_1-t_2)}}e^{iS_{\rm cl}/\hbar}\,, \tag{90}$$

$$S_{\rm cl} = \frac{m\omega}{2\sin\omega(t_1-t_2)}[(x_1^2+x_2^2)\cos\omega(t_1-t_2)-2x_1x_2]\,, \tag{91}$$

we can obtain the analytic solution of the TFD-type Schrödinger equation in thermal equilibrium as

$$\Psi_{\rm eq}(x,\tilde{x}) = \frac{1}{Z(\beta)^{1/2}}K(x,-i\hbar\beta/2;\tilde{x},0)\,, \tag{92}$$

$$Z(\beta) = \frac{1}{2\sinh(\beta\hbar\omega/2)}\,, \tag{93}$$

$$K(x,\tilde{x};-i\hbar\beta/2,0) = \sqrt{\frac{m\omega}{2\pi\hbar\sinh(\beta\hbar\omega/2)}}e^{R_{\rm eq}}\,, \tag{94}$$

$$R_{\rm eq} = -\frac{m\omega}{\hbar}\frac{(x^2+\tilde{x}^2)\cosh(\beta\hbar\omega/2)-2x\tilde{x}}{2\sinh(\beta\hbar\omega/2)}\,. \tag{95}$$

Then, the drift terms in the stochastic equations are given by

$$b_{\rm eq}(x(t),\tilde{x}(t)) = -\omega\left(x(t)\frac{\cosh(\beta\hbar\omega/2)}{\sinh(\beta\hbar\omega/2)}-\tilde{x}(t)\frac{1}{\sinh(\beta\hbar\omega/2)}\right)\,, \tag{96}$$

$$\tilde{b}_{{\rm eq}*}(x(t),\tilde{x}(t)) = \omega\left(\tilde{x}(t)\frac{\cosh(\beta\hbar\omega/2)}{\sinh(\beta\hbar\omega/2)}-x(t)\frac{1}{\sinh(\beta\hbar\omega/2)}\right)\,. \tag{97}$$

Note that the drift term $b_{\mathrm{eq}}(x(t), \tilde{x}(t))$ has the temperature dependence because of the temperature dependence of the wave function $\Psi_{\mathrm{eq}}(x, \tilde{x})$ and the thermal fluctuation of the non-tilde particle is induced through the quantum correlations with the tilde particle $\tilde{x}(t)$. In the limit of zero temperature $\beta \to \infty$, the drift term $b_{\mathrm{eq}}(x(t), \tilde{x}(t))$ is reduced to $b_{\mathrm{eq}}(x(t), \tilde{x}(t)) \to -\omega x(t)$, that is, the correlation between the non-tilde and tilde particles disappears. This way, our formalism reproduces the original Nelson's stochastic quantum mechanics at zero temperature. The numerical results of the non-tilde and tilde stochastic equations are shown in figures 1 and 2. Figure 1 shows stronger correlations between X and $\tilde{X}$ for higher temperature, namely larger thermal fluctuations. One can see that the probability distribution from numerical calculations of sample paths fits the analytic one well.

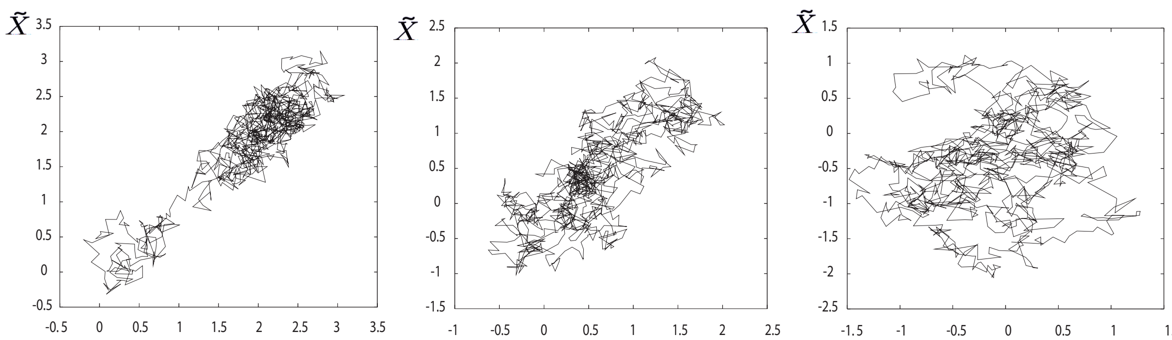

Figure 1. The typical sample path on $(X, \tilde{X})$ plane for $\bar{\beta} = 0.5$, $\bar{\beta} = 1$ and $\bar{\beta} = 3$ with $\bar{\beta} = \hbar\omega\beta$ and $X = \sqrt{\frac{m\omega}{\hbar}}x$.

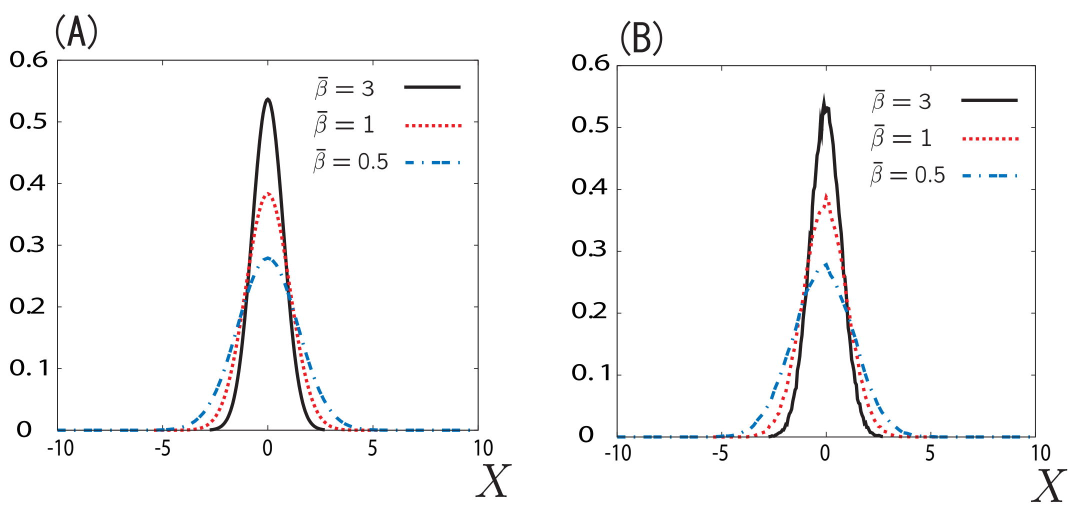

Figure 2. Figure (A): The analytic solutions of the probability distribution for $\bar{\beta} = 3$, $\bar{\beta} = 1$ and $\bar{\beta} = 0.5$ with $\bar{\beta} = \hbar\omega\beta$ and $X = \sqrt{\frac{m\omega}{\hbar}}x$. Figure (B): The probability distribution which is obtained numerically from 10^5 sample paths of the non-tilde and tilde stochastic equations.

Finally, we show the position-momentum uncertainty relation in stochastic mechanics [20]. Taking the sample average of equation (71), we obtain the relation: $E[\boldsymbol{p}] = E[\boldsymbol{p}_*]$ where we introduce the notation $\boldsymbol{p} = m\boldsymbol{b}$ and $\boldsymbol{p}_* = m\boldsymbol{b}_*$. On the other hand, multiplying equation (71) x_i, we can drive the relation: $E[x_i p_j - x_i p_{*j}] = -\hbar\delta_{ij}$. Utilizing the above relations, we can evaluate the quantity:

$$E\left[(x_i - E[x_i])\left(\frac{p_i - p_{*i}}{2} - E\left[\frac{p_i - p_{*i}}{2}\right]\right)\right] = -\frac{\hbar}{2}\,. \tag{98}$$

Applying the Schwarz's inequality to equation (98), the position-momentum uncertainty relation in stochastic mechanics is given by

$$\sqrt{\mathrm{Var[x_i]}}\sqrt{\mathrm{Var}[(p_i - p_{*i})/2]} \geq \frac{\hbar}{2}\,. \tag{99}$$

Now, we calculate the position and momentum uncertainty at finite temperature for harmonic oscillator. From a simple calculation, we obtain the variance of x_i and $(p_i - p_{*i})/2$ as

$$\mathrm{Var[x_i]} = \frac{\hbar}{2m\omega}\frac{1}{\tanh\left(\frac{\beta\hbar\omega}{2}\right)}\,, \tag{100}$$

$$\mathrm{Var}[(p_i - p_{*i})/2] = \frac{m\hbar\omega}{2}\frac{1}{\tanh\left(\frac{\beta\hbar\omega}{2}\right)}\,. \tag{101}$$

Then, the position and momentum uncertainty is given by

$$\sqrt{\mathrm{Var[x_i]}}\sqrt{\mathrm{Var}[(p_i - p_{*i})/2]} = \frac{\hbar}{2} + \hbar n\,, \tag{102}$$

$$n = \frac{1}{e^{\beta\hbar\omega} - 1}\,. \tag{103}$$

Note that the position and momentum uncertainty depend not only on Planck constant $\hbar$ but also on the temperature of the system. This result is consistent with the position-momentum uncertainty relation derived by thermal Bogoliubov transformation [14, 15].

6. Summary

In this paper, we extended the Nelson's stochastic quantum mechanics to thermal situation in framework of Thermo Field Dynamics (TFD) [16]. We set the four stochastic equations: two are for the non-tilde particle in forward and backward times, and the other are for the tilde particle in forward and backward times. The essence of our successful formulation is to make the pairs, i.e., the pair of non-tilde operator in forward time and tilde one in backward one, and the pair of non-tilde operator in backward time and tilde one in forward one, taking into account that the time-evolution of the tilde system corresponds to the time reversed one of the non-tilde one. We have shown that the four stochastic and two Nelson-Newton equations reproduce the corresponding TFD-type Schrödinger equation, which is equivalent to the Liouville-von Neumann equation. In our formalism, the drift terms of the stochastic equation have the temperature dependence and the thermal fluctuation is induced through the quantum

correlation between the non-tilde and tilde systems. In the limit of zero temperature, the temperature dependence in the drift terms disappears and our theory becomes the original Nelson quantum mechanics at zero temperature. Note that our formulation can be easily extended to many-body systems including systems of identical particles [8, 9]. In application of our theory to harmonic oscillator, we have analyzed the position-momentum uncertainty relation at finite temperature. Our combined formulation of the Nelson's stochastic quantum mechanics and TFD gives us a new insight into quantum theory in thermal situation.

Finally we comment on another view about the thermal and quantum fluctuations in our formalism. The origin of the thermal fluctuation in our formalism is quite different from that in the phenomenological theory of irreversible processes which is described by the classical Langevin equation with the temperature dependent random noise. Instead of the coordinates x and $\tilde{x}$, we consider new coordinates for harmonic oscillator as

$$X(t) = \sqrt{(1+n)}x(t) - \sqrt{n}\tilde{x}(t)\,, \tag{104}$$

$$\tilde{X}(t) = \sqrt{(1+n)}\tilde{x}(t) - \sqrt{n}x(t)\,, \tag{105}$$

$$n = \frac{1}{e^{\beta\hbar\omega} - 1}\,, \tag{106}$$

the stochastic equations for $X(t)$ and $\tilde{X}(t)$ are given by

$$dX(t) = -\omega X(t)dt + \sqrt{\frac{\hbar(1+n)}{m}}dW - \sqrt{\frac{\hbar n}{m}}d\tilde{W}\,, \tag{107}$$

$$d\tilde{X}(t) = \omega\tilde{X}(t)dt + \sqrt{\frac{\hbar(1+n)}{m}}d\tilde{W} - \sqrt{\frac{\hbar n}{m}}dW\,. \tag{108}$$

The drift terms of $X(t)$ and $\tilde{X}(t)$ have no temperature dependence, but the random noise terms depend on both the temperature and the Planck constant $\hbar$. In the zero temperature limit $\beta \to \infty$, the random noise terms approach standard Wiener processes with coefficient $\sqrt{\frac{\hbar}{m}}$. In high temperature limit, the $\hbar$ dependence of the random noise terms vanishes, as it is expected that the system becomes classical, and the random noise becomes $\sqrt{2kT/(\omega m)}(dW - d\tilde{W})/\sqrt{2}$. This way the stochastic equation for $X(t)$ becomes the classical Langevin equation.

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